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Abstract

Minimum variance portfolios constructed via mathematical optimization allow to significantly reduce equity risk. Minimum variance optimization, though bearing a certain level of complexity, cannot be regarded anymore as a "black box" process. We will argue in this paper that properly designed minimum variance investment approach can be relevant for investors seeking risk-efficient passive equity allocation, that it can be made transparent to investors by giving them more disclosure and intuition on the resulting optimized allocation and the choices made in by the optimization procedure. We discuss here how the minimum variance engine works and how to structure the optimization procedure in order to keep the minimum variance portfolio diversified.

Towards risk efficiency in passive investing

Since Harry Markowitz seminal work (1952) the investment community embraced the concept of optimal portfolio. While simple in its objective - delivering the best risk/return trade-off - building such a portfolio is a daunting task requiring implicit or explicit assumptions about future risk and returns. As a consequence, the resulting allocation remains optimal only *ex-ante*, and its *ex-post* performance is seriously compromised by forecasting errors.

Since the 1960s, the capital asset pricing model (CAPM) brought an elegant solution to the optimization problem, arguing that the most efficient portfolio is necessarily a broad market portfolio weighted by market capitalization of stocks. This corresponds to performing mean-variance optimization with market-implied forecasts of risk and returns. Theoretically backed by the powerful efficient market framework, this idea gained an enormous influence and reshaped the equity investing industry over the past 35 years. Mean-variance framework suggests another definition of passive equity investing: an investment process aiming at providing an access to the equity market or one of its segments, that does not require return forecasts for the stocks inside the chosen universe. The objective of "new passive" approach is to shape the portfolio in such a way that the equity exposure gives the best possible risk-adjusted performance.

Market-capitalization weighted portfolio is actually an "active" portfolio in the mean-variance sense. It is reliant on implicit future return forecasts contained in the stocks relative capitalization and is optimal only if this forecast is the best that one can have. There is only one optimal portfolio on the mean-variance frontier that is truly "passive" in its objective, that is the Minimum Variance Portfolio

(MVP). MVP is an optimal portfolio that is constructed by minimizing portfolio variance. The minimum variance construction does not use stocks' expected returns as inputs, and relies only on the covariance matrix.

Usually the MVP is depicted as the outmost left point on the mean-variance frontier, the less risky and the less performant one. But this picture is very misleading, since it is conditional on some non-homogeneous forecast of future returns. If such a forecast happens to be wrong, the ex-post performance of the MVP might be well above the ex-ante optimal portfolios that have higher risk. If ex-post the stocks will have similar returns, the MVP will take the place of the efficient tangent portfolio, giving the highest Sharpe ratio.

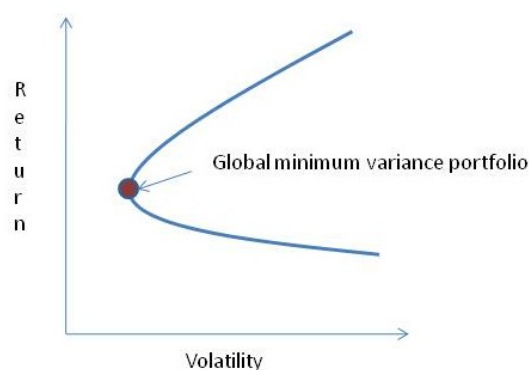


Figure 1: *Widespread but often misleading illustration of the Minimum Variance Portfolio*

There is a growing body of evidence on performance and characteristics of the "new passive" minimum variance investing. Over the past three years MSCI and DAX have created minimum variance versions of their benchmark equity indices. In addition, a growing amount of academic research is dedicated to minimum variance portfolios, where usually some test MVP are studied. All these portfolios share some important properties, as reduced ex-post

portfolio volatility with respect to the market-capitalization benchmarks. But the differences in portfolio construction across different minimum variance strategies can be significant¹. This stems from the fact that the Minimum variance methodology is not unique, and it is difficult for investors to get insight on the implications of minimum variance portfolio choice and the resulting stock allocation.

Technically, to construct a minimum-variance portfolio one needs a forecast of the covariance matrix and a mean-variance optimization engine. The latter is a mathematical procedure that decides which stocks to pick and what weights to give them to obtain the lowest possible volatility of the overall portfolio. The mechanics of this choice, especially in realistic cases where different portfolio constraints are involved, is by no means intuitive for investors. The widely spread belief that such a technique tends to pick low-volatility or low-beta stocks is relevant but far from fully describing the resulting allocation of a minimum variance portfolio.

We will argue in this paper that properly designed minimum variance investment approach can be relevant for investors seeking risk-efficient passive equity allocation, that it can be made transparent to investors by giving them more disclosure and intuition on the resulting portfolios and the choices made in constructing the mean-variance procedures, and, finally, that this approach can be consistently applied in a systematic investment mode.

¹According to the respective index guides, MSCI Minimum Volatility US index portfolio is forced to contain well above 60 stocks, while the DAXplus Minimum Variance US index is limited to maximum 50 stocks.

Reducing portfolio variance is a feasible goal

A natural question that investors ask is: is it possible to maintain a full exposure to the equity market while mitigating risk? An accurate view on the future risk structure is very important here. Risk profiles of single stocks as well as the structure of stocks comovements are resumed in the variance-covariance matrix. Minimum variance technique combines the information on expected risk levels to build an allocation with the minimum expected risk possible.

The dominant approach is to infer the future stocks variances and covariances from historical data. While subject to estimation errors, the historical method gives a relevant short and medium-term picture of the future risk. Stock volatility levels are highly persistent², and the covariance structure is persistently dominated by the broad market risk factor³.

Once the estimation of covariance matrix is built⁴, various constraints, that we will discuss below in greater detail, allow to adapt the technique to specific investor situations (in particular, limit short sales) and incorporate ex-ante risk management features in the portfolio (e.g. limiting concentration on single stocks).

Growing evidence, based on live records of minimum variance products and historical sim-

²This stems from the well-documented stylized facts about stocks returns (see e.g. Cont, 2001), such as persistence in absolute return values and volatility clustering.

³The dominance of the first eigenvalue of correlation matrix is discussed for example in the Bouchaud and Potters (2005)

⁴There are various techniques of error reduction in covariance estimation at one's disposal: the examples are covariance shrinkage (see e.g. Ledoit, Wolf, 2003), use of risk factor models (the example is the MSCI Barra risk estimation framework), or noise filtering with the random matrix theory (For a recent review see Bouchaud, 2009).

ulations, shows that volatility reduction using minimum variance technique is significant⁵.

The other question that is on everyone's mind is whether this new portfolio will perform better than the market-cap weighted benchmark. Is there a monetary advantage on concentrating solely on the least volatile equity configurations? This question is out of the subject of this paper, since we do not want to delve into estimating future relative or absolute returns, that is a huge and controversial subject on its own. The only point that we want to mention here is that historical evidence did not support the CAPM conjecture of performance reduction for low-volatility stocks with respect to high-risk stocks⁶. On the other hand, there is no clear understanding of why the low-risk stocks could have outperformed the market.

When we incorporate estimated stock returns into Markowitz optimization, the efficiency of the resulting allocation will be largely determined by the structure of these estimates and errors contained in it⁷, and to much lesser extent by the design of the optimization problem. Instead, when assuming homogeneous return estimates, the design of the optimization problem plays much greater role, and allows to build an allocation that will maintain its promise: exhibiting lower risk in the future while giving full exposure to equity market.

So, what happens inside the minimum variance optimizer?

Closer look at minimum variance optimization

We discuss here, step-by-step, the main ingredients of the minimum variance optimization problem. At each step of the discussion we describe the impact that each new element of optimization design has on the optimized allocation.

First we need to introduce some notations: we refer to σ as single-stock volatility, ρ - a correlation between two stocks and Σ - the covariance matrix that groups single-stock volatilities and pairwise correlations. Stocks' weights we denote by w . Now we can formulate minimum variance objective as follows (in matrix notations):

$$\min_w \bar{w}' \Sigma \bar{w}, \quad (1)$$

The above is a minimization problem, that is quadratic in the weights w . Standalone, this problem gives an empty solution. Since no return expectations are given, the natural choice is to invest 100% of the portfolio in cash ($w^* = 0$), shrinking the portfolio variance to the absolute zero.

The minimum variance mechanism needs additional parts to make the approach work. These parts are formulated as portfolio constraints, that we will add one by one to show their role.

Budget constraint:

$$\sum w_i = X\%$$

This constraint forces the optimizer to produce a non-empty solution, with X% net long investing in stocks. All existing minimum variance strategies employ it, usually with X = 100%. Thus the budget constraint is considered to be inherent to the minimum variance

⁵For example, the simulated MSCI Minimum Volatility World index had around 30% reduction in volatility with respect to the market-capitalization weighted index over the period 1995-2007, as documented by MSCI BARRA Research (2008)

⁶See for example A. Ang et al (2006)

⁷see for example Chorpá, Ziemba 1993

optimization problem, and this setup is often referred to as unconstrained.

In the general case the result is a long-short portfolio, where long investment exceeds the short side by the specified amount of X%. The long side of the portfolio will tend to be populated by less risky stocks and the short by more risky ones.

Exact long-short configuration will depend on the structure of the covariance matrix. For example, if the stocks are all uncorrelated ($\rho = 0$), there will be no short positions since all stocks will have a potential to provide additional risk reduction by diversification. There is another simple particular case of correlation structure that demonstrates well the stock selection: the case of equal pairwise correlations. We denote different stock volatilities as σ_i , where i goes from 1 to N , the number of stocks in the group. The pairwise correlation, that we assume to be equal, we denote by ρ .⁸

A useful relation that follows from the exact optimization solution in this case is:

$$\sigma_i < H(\sigma, \rho, N) \Rightarrow w_i > 0 \quad (2)$$

$$\sigma_i > H(\sigma, \rho, N) \Rightarrow w_i < 0 \quad (3)$$

$$\text{with } H(\sigma, \rho, N) \equiv H(\sigma) \left(1 + \frac{1}{N\rho} - \frac{1}{N} \right)$$

where $H(\sigma) = N(\sum_{i=1}^N 1/\sigma_i)^{-1}$, a harmonic average of the volatilities, and $H(\sigma, \rho, N) \geq H(\sigma)$ is a modified harmonic average, that is adjusted upwards by a function of correlation. This relation allows to infer which stocks will be assigned to the long portfolio side. Volatility of each stock will be compared to the threshold (represented by the modified harmonic average). The above formula covers also the case of uncorrelated stocks, where the threshold is shifted to infinity and all the stocks

⁸This situation is a very rough approximation of the equity market, where normally the correlations among stocks are positive and significant.

are assigned to the long side. For another extreme case $\rho \rightarrow 1$, the group is divided exactly by the harmonic average of volatilities. When the correlation ρ is growing from one extreme (0) to the other (1), the optimizer is assigning more and more stocks to the short portfolio starting from the more volatile ones, until it transfers there around 50% of the group (the exact percentage depends on where the harmonic average is situated).⁹

Another important piece of information on the resulting allocation is the form of the weights assigned to the stocks. The weights will be inversely proportional to the stocks' variance, and an exact formula is known in this case.¹⁰

The situation is more complex in the general case of non-equal pairwise correlations, and the optimizer will add to the portfolio some stocks with consistent volatility that appear weakly correlated to the rest of the selected group.

The long-short allocation gives interesting insights on the mechanics of the minimum variance optimizer, but is hardly suitable for investors. Note, that single stock weights are not restricted and nothing prevents them from assuming very high absolute values.

The next step is crucial to adopt the minimum variance process to the needs of a long-only investor.

Short sale constraint:

$$w_i \geq 0$$

⁹Harmonic average is always smaller than arithmetic average. The distance between the two is related to the standard deviation of the sample as well as to the outliers.

¹⁰The general solution in the matrix form is $w = \frac{\Sigma^{-1} \vec{u}}{\vec{u}' \Sigma \vec{u}}$, where \vec{u} is a vector of ones. In the case of constant correlation the formula becomes: $w_i = \frac{1}{\sigma_{MV}} \left[\frac{1}{\sigma_i^2} - \frac{\rho}{1+(N-1)\rho} \sum_j \frac{1}{\sigma_j} \right] = \frac{1}{\sigma_{MV}} \frac{1}{\sigma_i^2} \left[1 - \frac{\sigma_i}{H(\sigma, \rho, N)} \right]$, where σ_{MV} is the portfolio variance.

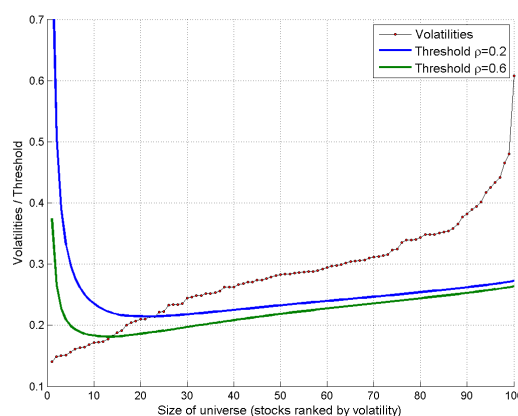
This is a constraint that all the existing "passive" minimum variance strategies have. Its role is critical, since this constraint forces the optimizer to assign only non-negative positions to stocks, and actually it dictates the optimizer to restrict the portfolio on the "zero-weight boundary", often by excluding a great amount of the initial universe from the portfolio.

In this case no close-form solution is available for analysis in the constant correlation case, but actually a modified version of the above relation holds. The optimizer implicitly defines a certain threshold (which is different from the long-short case above), that decides whether a certain stock will receive zero weight or not. The position of this threshold determines how many stocks will be included in the portfolio.

One can imitate the optimization by picking the stocks one by one, starting from the least volatile ones, and comparing their volatility to the modified harmonic average of the stocks **already admitted to the portfolio**. If the new stock has volatility below the threshold, it will be added to the portfolio, otherwise the procedure stops and no new stocks are added. This works like a test on remaining diversification potential of adding more stocks, and the threshold in this procedure is changing after adding a new stock in the portfolio.

Below we give a graphical representation of the stock selection process:

This example takes a universe of 100 stocks with volatilities that correspond to recent 6-months volatilities of the 100 largest US stocks, and with constant pairwise correlations. Several moving threshold are constructed using the modified harmonic average introduced above, and assuming different correlation levels. To construct each moving threshold the formula is applied progressively to growing sets of stocks, sorted by their volatility in descending order. All the stocks with volatility situated below the threshold will be selected in the



optimized portfolio. The example shows that from 10 to 20 stocks among the 100 largest US stocks will be selected by the optimizer, depending on correlation level. This means that only a small part of the initial group passes the selection. This result is somewhat disappointing, since the portfolio is very concentrated. It gets worse when one looks inside this small portfolio and discovers that the three least volatile stocks got a combined weight of 40% of the portfolio.

One of the problems leading to the extreme concentration is the presence of a certain amount of estimation error. The minimum variance approach appears to be not sensitive to the overestimation of volatility ("tail" events), but instead suffers from the underestimation of risk that happens for some stocks. Indeed, the harmonic average of volatilities that appeared in above examples is sensitive to the presence of very low values in the sample. As we mentioned already, there are several methods to improve the situation. One consists in applying various techniques that reduce the estimation noise and improve the stability of the estimated variance (e.g. shrinking approaches or variance estimation using factor models). The other way out is to introduce further constraints on portfolio weights in the optimization problem.

Additional Constraints

This is the point where additional constraints are brought onto the stage to fix the concentration problem and guarantee various risk management and allocation restrictions that an investor might have. The most important examples are linear constraints:

1. limit the maximal weight of one stock,
2. ensure a certain minimum weight for each stock in the universe,
3. limit single sector or country exposures,
4. constrain possible sector/country exposures around broad market exposures...

The exact combination of these additional constraints is what makes the difference among various minimum variance indices and products. Still, the mechanics of the optimizer remain largely untouched by the new constraints. It was shown¹¹ that application of constraints can be translated into certain transformations of the initial covariance matrix. In particular, an optimization problem with no short sale constraint ($w_i \geq 0$) that uses a covariance matrix Σ is identical to the traditional unconstrained optimization problem with a shrunk covariance matrix.¹² There are shrinkage transformations that correspond to other (linear) constraints, for example maximal weight constraint¹³.

¹¹see Jagannathan and Ma (2003).

¹² The new covariance matrix can be given as $\tilde{\Sigma} = \Sigma - \tilde{\lambda} \tilde{u} - \tilde{u} \tilde{\lambda}$, where $\tilde{\lambda}$ is the vector of Lagrange multipliers found in the constrained optimization. This shrinkage corresponds to reducing outliers in the matrix.

¹³see e.g. Roncalli, 2010

How to keep minimum variance portfolio diversified

How do the weight bounds impact the portfolio efficiency, and do they allow to achieve the desired diversification? For example, to ensure that a certain number of stocks will be present in the portfolio, one can set a maximal weight per stock w_{max} , such as $1/w_{max} \sim K$, where K is the minimal number of stocks that will be present in the portfolio. But from the point of view of covariance shrinkage, the impact of linear constraints on covariance matrix can be quite substantial, since the resulting allocation could have a lot of weights that lie on the w_{max} bound. This is somewhat worrying, given that the empirical covariance has predictive power of the future level of risk. In this regard, portfolio optimization design that has a smooth impact on covariance matrix and preserves the relative riskiness of the stocks will be preferable.

One very interesting solution comes from the idea to apply quadratic, instead of linear, constraints on the optimal portfolio. The new constraints were first proposed by DeMiguel et al (2009) under the name of 2-norm constraints, and shown to have several nice properties. Here we consider a quadratic equality constraint, that is formulated as:

$$\vec{w}' \vec{w} = \sum_i w_i^2 = \frac{1}{H} \quad (4)$$

The impact of this constraint on the covariance matrix is quite intuitive: it corresponds to a classical covariance shrinkage towards an identity matrix. Indeed, adding the new constraint to the objective function with a Lagrange multiplier, one sees that the shrunk covariance matrix is simply

$$\tilde{\Sigma} = \Sigma + \lambda Id \quad (5)$$

This shrinkage method is used in the James-Stein estimator¹⁴. The target shrinkage matrix - the identity matrix, is biased with respect to the "true" covariance, but allows to efficiently reduce the estimation noise. The degree of shrinkage depends on the parameter λ , that comes from the solution of the constrained optimization problem. The bigger the value of λ , the stronger is the impact of the constraint. In the limit of $\lambda \rightarrow \infty$ the shrunked covariance goes towards the identity matrix, that implies a very simple solution for the optimization problem: an equi-weighted portfolio (if compatible to all the additional constraints in the problem). So, the higher the Lagrange multiplier, the closer the solution is to the equal weights.

The degree of shrinkage will depend on the value of the constraint target appearing on the right-hand side. One can note that the form of this constraint is familiar: the sum of squared weights corresponds to the Herfindahl-Hirschman index (HHI), that measures the concentration of firms in the industries, to assess the degree of competition. The bigger the value of HHI, the higher is the concentration. This analogy suggests similar interpretation of the quadratic constraint in the optimization problem: it merely represents a diversification target in terms of portfolio concentration. It is very easy to induce a minimum number of stocks that will be held in a portfolio that has a certain diversification (HHI) level. If the diversification target is set, for example, at $\frac{1}{H} = 0.02$, then the portfolio cannot contain less than $1/0.02 = 50$ stocks.¹⁵

This diversification target approach provides a way to ensure meaningful portfolio diversification in a way that the covariance matrix structure is modified smoothly. Integrating

this constraint in the minimum variance optimization problem, allows to design a systematic strategy that has a desired diversification level, without saturating a maximal weight constraint.

Conclusion

We considered here the main ingredients of the minimum-variance equity investing. This "new passive" systematic approach does not rely on forecasts of stock returns and uses as input only the covariance matrix. The portfolio resulting from minimum-variance optimization has the smallest *ex-ante* volatility, and exhibits a significant reduction in *ex-post* risk as well, with respect to the market-capitalization-weighted benchmark.

The portfolio optimization problem behind the minimum-variance portfolio is not uniquely defined. Different constraints can be integrated into the optimization setup, that could change significantly the properties of the resulting optimal allocation. The most important constraints, such as short sale ban, can be combined with restrictions on the maximal possible weight for single stock, as well as different constraints on groups of stocks (for example, on the sector level). The constraints help to control the composition of minimum variance portfolio, setting up risk-management limits that ensure certain level of diversification of factor exposure. However the constraints necessarily transform the information contained in the covariance matrix, and over-constrained configurations could have very few information from the initial covariance matrix left.

An alternative to the use of bounding linear constraints is to ensure sufficient diversification via targeting a certain portfolio 2-norm, that is a sum of squared portfolio weights. This constraint ensures that a certain mini-

¹⁴See James, Stein (1961).

¹⁵It is easier to formulate the diversification target directly in terms of number of stocks, i.e. specifying $H = 50$ rather than $\frac{1}{H} = 0.02$.

mum number of stocks will be present in the portfolio, this minimum number of stocks being just the inverse of the portfolio norm target. This diversification target constraint induces a smooth transformation of initial covariance matrix, that corresponds to James-Stein covariance shrinkage with an identity matrix.

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